

# LIMIT DIRECTIONS OF A VECTOR COCYCLE, REMARKS AND EXAMPLES

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**ABSTRACT.** We study the set  $\mathcal{D}(\Phi)$  of limit directions of a vector cocycle  $(\Phi_n)$  over a dynamical system, i.e., the set of limit values of  $\Phi_n(x)/\|\Phi_n(x)\|$  along subsequences such that  $\|\Phi_n(x)\|$  tends to  $\infty$ . This notion is natural in geometrical models of dynamical systems where the phase space is fibred over a basis with fibers isomorphic to  $\mathbb{R}^d$ , like systems associated to the billiard in the plane with periodic obstacles. It has a meaning for transient or recurrent cocycles.

Our aim is to present some results in a general context as well as for specific models for which the set of limit directions can be described. In particular we study the related question of sojourn in cones of the cocycle when the invariance principle is satisfied.

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## INTRODUCTION

Let  $(X, \mu, T)$  be an ergodic dynamical system and  $\Phi$  be a measurable function on  $X$  with values in  $\mathbb{R}^d$ . The ergodic sums  $\Phi_n(x) := \sum_{k=0}^{n-1} \Phi(T^k x)$ ,  $n \geq 1$ , define a vector process. When  $\Phi$  is integrable and not centered, this process tends a.s. to  $\infty$  in the direction of the mean  $\int \Phi d\mu$ . A general question, when  $\Phi$  is centered or for a measurable non integrable  $\Phi$ , is to find in which directions at infinity the ergodic sums are going. The set of these directions is a kind of boundary for the cocycle  $(\Phi_n)$ , i.e. for the process of ergodic sums.

This leads to the notion of limit directions and to the cohomologically invariant notion of essential limit directions. The limit directions of a vector cocycle  $(\Phi_n)$  over a dynamical system can be defined as the limit values of  $\Phi_n(x)/\|\Phi_n(x)\|$  along subsequences such that  $\|\Phi_n(x)\|$  tends to  $\infty$ .

The notion of limit directions is natural in geometrical models of dynamical systems where the phase space is fibred over a basis with fibers isomorphic to  $\mathbb{R}^d$ , like the dynamical systems associated to the billiard in the plane with periodic obstacles. It has a meaning for recurrent cocycles as well as for transient cocycles.

Our aim is to present some results in a general context (Section 2) and for specific models where the set of limit directions can be made explicit. In Subsection 2.5 1-dimensional cocycles are considered and some classical results are recalled or slightly extended.

In Section 3.1, we apply properties like the CLT for subsequences or the invariance principle to study essential limit directions and the behavior of the process induced by the cocycle on the sphere. For  $d \geq 2$ , when  $\Phi$  satisfies a Central Limit Theorem, one can think that the limit behavior of the sums is analogous to that of a Brownian motion, in particular in terms of visit of cones. In the last subsection 3.2 this is shown to be the case, at least if  $\Phi$  satisfies Donsker's invariance principle.

## 1. Preliminaries

Let  $(X, \mathcal{B}, \mu, T)$  be a dynamical system, where  $(X, \mathcal{B})$  is a standard Borel space,  $T$  an invertible measurable map  $T : X \rightarrow X$  and  $\mu$  a probability measure which is  $T$ -invariant. Let  $\Phi$  be a measurable function on  $X$  with values in  $G = \mathbb{R}^d$ . The process  $(\Phi \circ T^n)_{n \geq 1}$  is stationary. Recall that, under general assumptions, every stationary process  $(X_n)$  with values in  $\mathbb{R}^d$  can be represented in such a way for some dynamical system and some measurable  $\Phi$ .

Part of the results below are valid when  $\mu$  is only supposed to be a  $\sigma$ -finite  $T$ -quasi-invariant measure such that  $T$  is *conservative* for  $\mu$ , i.e., for every measurable  $B$  in  $X$ , for  $\mu$ -a.e.  $x \in B$ , there is  $n(x) > 0$  such that  $T^{n(x)}x \in B$ . Nevertheless for the sake of simplicity, excepted in Subsection 2.5, we will restrict the presentation to the framework of a probability invariant measure  $\mu$ .

Excepted in Section 2.5, the system  $(X, \mu, T)$  is supposed to be ergodic. All equalities are understood to hold  $\mu$ -a.e. All sets that we consider are measurable and (unless the contrary is explicitly stated) with positive measure.

To  $\Phi$  is associated a *cocycle*  $(\Phi_n)_{n \in \mathbb{N}}$  defined by  $\Phi_0(x) = 0$ ,

$$\Phi_n(x) = \Phi(x) + \dots + \Phi(T^{n-1}x), \text{ for } n \geq 1, \text{ and } \Phi_n = -\Phi_{-n} \circ T^n, \text{ for } n < 0,$$

and a map  $T_\Phi$  (called *skew product*) acting on  $X \times \mathbb{R}^d$  by

$$(1) \quad T_\Phi : (x, y) \rightarrow (Tx, y + \Phi(x)).$$

The cocycle relation  $\Phi_{n+p}(x) = \Phi_n(x) + \Phi_p(T^n x)$ ,  $\forall n, p \in \mathbb{Z}$  is satisfied. The cocycle gives the position in the fiber after  $n$  iterations of  $T_\Phi$ :

$$T_\Phi^n(x, y) = (T^n x, y + \Phi_n(x)).$$

The cocycle  $(\Phi_n)$  can be viewed as a "*stationary*" walk in  $\mathbb{R}^d$  "driven" by the dynamical system  $(X, \mu, T)$ . It is also the sequence of ergodic sums of  $\Phi$  for the action of  $T$ . We will use as well the notation  $(\Phi, T)$ .

The Lebesgue measure on  $\mathbb{R}^d$  is denoted by  $m(dy)$  or simply  $dy$ . The map  $T_\Phi$  leaves invariant the measure  $\mu \times m$  denoted by  $\lambda$ .

Recall that a cocycle  $(\Phi_n)$  over  $(X, \mu, T)$  is *transient* if  $\lim_n \|\Phi_n(x)\| = +\infty$ , for a.e.  $x \in X$ . It is *recurrent* if  $\liminf_n \|\Phi_n(x)\| < \infty$ , for a.e.  $x \in X$ . It is well known that, when  $T$  is ergodic, a cocycle is either transient or recurrent (see the comment below).

Recurrence of the cocycle is equivalent to conservativity of the map  $T_\Phi$  for the measure  $\lambda$ . When  $(\Phi_n)$  is recurrent, then  $(\Phi_n(x))$  returns for a.e.  $x$  infinitely often in any neighborhood of the origin. In dimension 1, if  $\Phi$  is integrable and  $(X, \mu, T)$  is ergodic, then  $(\Phi_n)_{n \in \mathbb{Z}}$  is recurrent if and only if  $\mu(\Phi) = 0$ . In higher dimension, recurrence requires stronger assumptions.

## Induced map

Let us recall some definitions and notations about induced maps.

Let  $B$  be a measurable set of positive  $\mu$ -measure. On  $B$  equipped with the measure  $\mu_B = \mu(B)^{-1} \mu|_B$ , the induced transformation is  $T_B(x) = T^{R(x)}(x)$ , where  $R(x)$  is the return time  $R(x) = R_B(x) := \inf\{j \geq 1 : T^j x \in B\}$ . The return time is well defined for a.e.  $x \in B$  by conservativity of the system. We induce<sup>1</sup>  $\Phi$  on  $B$  by putting

$$(2) \quad \Phi^B(x) := \Phi_{R(x)}(x) = \sum_{j=0}^{R(x)-1} \Phi(T^j x).$$

The "induced" cocycle is  $\Phi_n^B(x) := \Phi^B(x) + \Phi^B(T_B x) \dots + \Phi^B(T_B^{n-1} x)$ , for  $n \geq 1$ .

If  $\Phi$  is recurrent, then each induced cocycle  $(\Phi_n^B)$  is recurrent. Indeed  $(T_B)_{\Phi_B}$  is the induced map on  $B \times G$  of  $T_\Phi$  which is conservative.

<sup>1</sup>In short the function  $\Phi$  itself will also be called "cocycle" and  $\Phi^B$  "induced cocycle" on  $B$ .

If  $T$  is ergodic, then  $(B, \mu_B, T_B)$  is ergodic. The converse is true when  $X = \bigcup_n T^n B$ .

When the map  $T$  is ergodic, the above formulas can be extended to  $X$  by setting, for every measurable set  $B$  of positive measure, for a.e.  $x \in X$ :

$$(3) \quad R_B(x) = \inf\{j \geq 1 : T^j x \in B\}, \quad \Phi^B(x) = \sum_{j=0}^{R_B(x)-1} \Phi(T^j x).$$

Recall that two  $\mathbb{R}^d$ -valued cocycles  $(\Phi^1, T)$  and  $(\Phi^2, T)$  over the dynamical system  $(X, \mu, T)$  are  $\mu$ -cohomologous with transfer function  $\Psi$ , if there is a measurable map  $\Psi : X \rightarrow \mathbb{R}^d$  such that

$$(4) \quad \Phi^1(x) = \Phi^2(x) + \Psi(Tx) - \Psi(x), \text{ a.e.}$$

$\Phi$  is a  $\mu$ -coboundary, if it is cohomologous to 0.

We choose a norm  $\|\cdot\|$  on  $\mathbb{R}^d$ . We will use the inequality

$$(5) \quad \|\Phi_{n+1}(x)\| - \|\Phi_n(Tx)\| \leq \|\Phi(x)\|.$$

## 2. Limit directions of a vector cocycle, general properties

### "0-1" properties for a cocycle

Let  $(\Phi_n)$  be a cocycle over an ergodic dynamical system  $(X, \mu, T)$ . Some of its limit properties are related to the ergodicity of the skew product  $T_\Phi$ . For example, equidistribution properties (comparison of the number of visits to sets of finite measure) are given by the ratio ergodic theorem when the skew product  $T_\Phi$  is ergodic.

There are also limit properties which do not a priori require ergodicity of the skew product, but appear as "0-1" properties, in the sense that either they are satisfied by a.e.  $x$ , or are not satisfied by a.e.  $x$ .

More precisely, let  $\mathcal{P}(x)$  be a property which, for  $x \in X$ , is satisfied or not by the sequence  $(\Phi_n(x))$ . If the set  $\mathcal{A}_\mathcal{P} := \{x : \mathcal{P}(x) \text{ is true}\}$  is measurable and invariant by the map  $T$ , then by ergodicity of  $(X, \mu, T)$  this set has measure 0 or 1: either  $\mathcal{P}(x)$  is true for a.e.  $x$ , or  $\mathcal{P}(x)$  is false for a.e.  $x$ .

Sometimes, for an asymptotic property  $\mathcal{P}$ , the set  $\mathcal{A}_\mathcal{P}$  can be described in term of lim sup of a sequence of sets and its invariance by the map  $T$  can easily be checked.

The dichotomy between *recurrence* and *transience* of a cocycle is an example of a "0-1" property: the property  $\mathcal{R}$  "the cocycle is recurrent" corresponds to the set  $\mathcal{A}_\mathcal{R} = \bigcup_{M \geq 1} \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n^M$ , where  $A_n^M = \{x : \|\Phi_n(x)\| \leq M\}$ .

Indeed, from the inequality (5) it follows  $T^{-1}\mathcal{A}_\mathcal{R} = \mathcal{A}_\mathcal{R}$ . Therefore, when  $(X, \mu, T)$  is ergodic, either for  $\mu$  a.e. every  $x$ ,  $\lim_n \|\Phi_n(x)\| = +\infty$ , or for a.e.  $x$  the cocycle  $(\Phi_n(x))$  returns infinitely often in some compact set depending on  $x$ . In the latter case, an argument based on Poincaré recurrence lemma implies that the cocycle returns to any neighborhood of 0, for a.e.  $x$ .

We give below another example: the notion of limit direction.

## 2.1. Limit directions.

### Essential values and regularity

First we recall the classical notion of essential values of a recurrent cocycle with values in an abelian lsc group  $G$  (cf. K. Schmidt [Sc77]). A point  $\infty$  is added to  $G$  with the natural notion of neighborhood. For our purpose, we restrict ourselves to the case  $G = \mathbb{R}^d$ .

**Definition 2.1.** An element  $a \in G \cup \{\infty\}$  is an *essential value* of the cocycle  $(\Phi, T)$  (with respect to  $\mu$ ) if, for every neighborhood  $V(a)$  of  $a$ , for every measurable subset  $B$  of positive measure,

$$(6) \quad \mu(B \cap T^{-n}B \cap \{x : \Phi_n(x) \in V(a)\}) > 0, \text{ for some } n \geq 0.$$

The property (6) can be stated in the equivalent way:

$$(7) \quad \mu(\{x \in B : \Phi_n^B(x) \in V(a)\}) > 0, \text{ for some } n \geq 0.$$

We denote by  $\overline{\mathcal{E}}(\Phi)$  the set of essential values of the cocycle  $(\Phi, T)$  and by  $\mathcal{E}(\Phi) = \overline{\mathcal{E}}(\Phi) \cap G$  the *set of finite essential values*.

Let us recall some facts. The set  $\mathcal{E}(\Phi)$  is a closed subgroup of  $G$ . A cocycle  $\Phi$  is a coboundary if and only if  $\overline{\mathcal{E}}(\Phi) = \{0\}$ . We have  $\mathcal{E}(\Phi \bmod \mathcal{E}(\Phi)) = \{0\}$ . Two cohomologous cocycles have the same set of essential values.

It is well known ([Sc77], [Aa97]) that the set  $\mathcal{E}(\Phi)$  coincides with  $\mathcal{P}(\Phi)$ , the group of periods  $p$  of the measurable  $T_\Phi$ -invariant functions on  $X \times G$ , i.e., the elements  $p \in G$  such that for every  $T_\Phi$ -invariant  $F$ ,  $F(x, y + p) = F(x, y)$ ,  $\lambda - a.e.$  This shows that  $\mathcal{E}(\Phi) = G$  if and only if  $(X \times G, \lambda_X, T_\Phi)$  is ergodic.

**Definition 2.2.** We say that the cocycle defined by  $\Phi$  is *regular*, if it is cohomologous to a cocycle which has values in a closed subgroup  $H$  of  $G$  and is ergodic on  $X \times H$ . The group  $H$  in the definition is  $\mathcal{E}(\Phi)$ .

Now we consider the notion of limit directions and essential limit directions. The cocycle can be recurrent or transient.

### Limit directions

For  $v \in \mathbb{R}^d \setminus \{0\}$ , let  $\tilde{v} := v/\|v\|$  be the corresponding unit vector in the unit sphere  $\mathbb{S}^{d-1}$ . For every  $\mathbb{R}^d$ -valued cocycle  $(\Phi_n)$ , we obtain a process (directional process)  $(\tilde{\Phi}_n)_{n \geq 1}$  with values in  $\mathbb{S}^{d-1}$  (defined outside the values  $(n, x)$  such that  $\Phi_n(x) = 0$ ).

**Definition 2.3.** A vector  $u$  is a *limit direction* of the cocycle  $(\Phi_n(x))$  at  $x$ , if there exists a subsequence  $(n_k(x))$  such that  $\|\Phi_{n_k}(x)\| \rightarrow \infty$  and  $\Phi_{n_k}(x)/\|\Phi_{n_k}(x)\|$  converges to  $u$ .

The subset for which the property  $\mathcal{P}_u$ : " $u$  is a limit direction of  $(\Phi_n(x))$ " holds is

$$(8) \quad \mathcal{A}(u) = \bigcap_{V, M} \bigcap_N \bigcup_{n \geq N} \{x \in X : \|\Phi_n(x)\| > M \text{ and } \Phi_n(x)/\|\Phi_n(x)\| \in V\}.$$

where the intersection is taken over a countable basis of neighborhoods  $V$  of  $u$  and the positive integers  $M$ .

The set of limit directions of the cocycle  $(\Phi_n(x))$  for  $x \in X$  is defined as

$$\mathcal{D}(\Phi)(x) := \{u : \exists (n_k(x)) : \|\Phi_{n_k(x)}(x)\| \rightarrow \infty \text{ and } \Phi_{n_k(x)}(x)/\|\Phi_{n_k(x)}(x)\| \rightarrow u\}.$$

**Remarks 1.** a) From (5) it follows that  $\mathcal{A}(u)$  is invariant by the map  $T$ , so that  $\mathcal{P}_u$  is a "0-1" property.

b) If  $\Phi$  is integrable and  $\int \Phi d\mu \neq 0$ , then by the ergodic theorem  $\mathcal{D}(\Phi)$  reduces to the direction defined by the mean of  $\Phi$ . Therefore, when  $\Phi$  is integrable, the interesting case is when  $\int \Phi d\mu = 0$ .

c) If  $\Phi$  is a coboundary,  $\Phi = \Psi - \Psi \circ T$ , then the set of limit directions can be deduced from the support of the law of  $\Psi$  in  $\mathbb{R}^d$ . When this law gives a positive measure to each cone truncated from the origin, then, by ergodicity of  $T$ , the set  $\mathcal{D}(\Phi)$  coincide with  $\mathbb{S}^{d-1}$ .

d) Billiards in the plane with periodic obstacles yield geometric examples of centered vector cocycles with a geometric interpretation of the limit directions (for these models, see for example [Pe00], [SzVa04] for the dispersive billiards, [Gu10], [CoGu12] for the billiards with polygonal obstacles).

**Lemma 2.4.** *There is a closed set  $\mathcal{D}(\Phi)$  such that  $\mathcal{D}(\Phi)(x) = \mathcal{D}(\Phi)$ , for a.e.  $x$ . It is empty if and only if  $\Phi$  is a coboundary:  $\Phi = \Psi - \Psi \circ T$ , with  $\Psi$  bounded.*

*Proof.* Clearly  $\mathcal{D}(\Phi)(x)$  is a closed subset of  $\mathbb{S}^{d-1}$ . The invariance  $\mathcal{D}(\Phi)(Tx) = \mathcal{D}(\Phi)(x)$  follows from (5). Using the Hausdorff distance on the set of closed subsets of  $\mathbb{S}^{d-1}$  and ergodicity, we obtain that  $\mathcal{D}(\Phi)(x)$  is a.e. equal to a fixed closed subset.

If  $\mathcal{D}(\Phi)$  is empty, then, for a.e.  $x$ , the sequence  $(\Phi_n(x))$  is bounded. This implies that there is a measurable function  $\Psi$  such that  $\Phi = \Psi - \Psi \circ T$ . By ergodicity of  $T$ ,  $\Psi$  is bounded. The converse is clear.  $\square$

**Definition 2.5.**  $\mathcal{D}(\Phi)$  will be called *set of limit directions* of  $(\Phi_n)$ . We write also  $\mathcal{D}(T, \Phi)$  instead of  $\mathcal{D}(\Phi)$  to explicit the dependence on  $T$ .

In other words, the "limit set"  $\mathcal{D}(\Phi)$  is in the transient case the attractor in the sphere  $\mathbb{S}^{d-1}$  of the process  $(\tilde{\Phi}_n)_{n \geq 1}$  introduced above.

In the last section, we will show that under a strong stochastic hypothesis, this process  $(\tilde{\Phi}_n)_{n \geq 1}$  visits any non empty open set in  $\mathbb{S}^{d-1}$  and stays there during longer and longer intervals of time. This property can be formalized as follows.

Let  $(Z_n)$  be a process defined on  $(X, \mu)$  with values in a metric space  $Y$ . A first question is about transitivity: does  $(Z_n)$  visit every non empty open set in  $Y$ .

With the previous notion of limit direction, for the process  $(\frac{\varphi_n(\cdot)}{\|\varphi_n(\cdot)\|})$  associated to a transient cocycle  $(\varphi_n)$ , this means  $\mathcal{D}(\varphi) = \mathbb{S}^{d-1}$ .

A stronger quantitative property is the following:

$$(9) \quad \limsup_n \frac{1}{n} \sum_1^n \mathbf{1}_V(Z_k(x)) = 1 \text{ a.e., for every non empty open set } V \text{ in } Y.$$

Clearly this property implies  $\liminf_n \frac{1}{n} \sum_1^n \mathbf{1}_V(Z_k(x)) = 0$  a.e., for every non empty open subset  $V$  in  $Y$  with a complement with non empty interior.

We will now discuss some general properties of the set of limit directions. The set of limit directions  $\mathcal{D}(\Phi_B, T^B)$  for the induced cocycle  $\Phi^B$  and the induced map  $T^B$  is denoted by  $\mathcal{D}(\Phi_B)$  or  $\mathcal{D}(B)$ .

We have the equivalence:

**Lemma 2.6.** *a) A cocycle is a coboundary if and only if there is  $B$  of positive measure such that the set of limit directions for the induced cocycle on  $B$  is empty.  
b) If  $\Phi$  and  $\Phi'$  are cohomologous, there is  $B$  such that the corresponding induced cocycles on  $B$  have the same set of limit directions.*

*Proof.* a) If  $u$  is a limit direction for  $(T_A, \Phi_A)$ , then it is also a limit direction for  $(T, \Phi)$ ; hence the inclusion  $\mathcal{D}(T_A, \Phi^A) \subset \mathcal{D}(T, \Phi)$ .

By a compactness argument on the set of directions, if  $(A_n)_{n \geq 1}$  is a sequence of decreasing sets with positive measure in  $X$ , then  $(\mathcal{D}(T_{A_n}, \Phi^{A_n}))_{n \geq 1}$  is decreasing and the intersection is non empty, except if  $\mathcal{D}(T_{A_{n_0}}, \Phi^{A_{n_0}})$  is empty for some  $n_0$ .

If  $(\Phi_n)$  is bounded, or equivalently if  $\Phi = \Psi - \Psi \circ T$ , with a bounded  $\Psi$ , then clearly  $\mathcal{D}(T, \Phi)$  is empty.

Suppose now that  $\Phi$  is a coboundary,  $\Phi = \Psi - \Psi \circ T$ , with  $\Psi$  measurable. Let  $B$  be a set such that  $\Psi$  is bounded on  $B$ . Then the induced cocycle  $(\Phi_n^B)_{n \geq 1}$  is bounded, since  $\Phi_n^B = \Psi - \Psi \circ T_B^n$ . Therefore  $\mathcal{D}(T_B, \Phi^B)$  is empty.

Conversely, if there is  $B$  of positive measure in  $X$  such that  $\mathcal{D}(T_B, \Phi^B)$  is empty, then the induced cocycle  $(\Phi_n^B)$  is bounded, so  $\Phi^B$  is a  $T_B$  cocycle. By Lemma 2.7 below,  $\Phi$  is a coboundary.

b) Let  $\Phi$  and  $\Phi'$  be such that  $\Phi' = T\Psi - \Psi + \Phi$  for a measurable  $\Psi$ . Let  $B$  such that  $\Psi$  is bounded on  $B$ . Then  $\Phi_n'^B = T_B^n \Psi - \Psi + \Phi_n^B$  with  $T_B^n \Psi - \Psi$  bounded, which implies that  $(\Phi_n'^B)$  and  $(\Phi_n^B)$  have the same limit directions.  $\square$

**Lemma 2.7.** *Let  $B$  be such that  $X = \cup_{k \geq 0} T^k B$ . If  $\Phi^B$  is a  $T_B$ -coboundary, then  $\Phi$  is a  $T$ -coboundary.*

*Proof.* For  $\mu$ -a.a.  $y \in X$  there are a unique  $x \in B$  and an integer  $k$ ,  $0 \leq k < R_B(x)$ , such that  $y = T^k x$ . Suppose that there is  $\Psi$  on  $B$  such that:  $\Phi^B = \Psi - \Psi \circ T^B$ . We define  $\zeta$  on  $X$  by taking, for  $0 \leq k < R_B(x)$ ,  $\zeta(T^k x) = \Psi(x) - \Phi_k(x)$ .

We have  $\Phi(y) = \zeta(y) - \zeta(Ty)$ . Indeed, for  $y = T^k x$ ,  $0 \leq k < R_B(x) - 1$ , the relation is satisfied by construction; for  $y = T^k x$  with  $k = R_B(x) - 1$ , the relation follows from the coboundary relation for the induced cocycle.  $\square$

Now let us show that two sets  $A_1$  and  $A_2$  have always non disjoint sets of limit directions, unless  $\Phi$  is a coboundary.

**Lemma 2.8.** *For any two sets  $A_1$  and  $A_2$ , there is  $B_1 \subset A_1$  such that  $\mathcal{D}(B_1) \subset \mathcal{D}(A_2)$ .*

*Proof.* Let  $B_1 \subset A_1$  be such that  $\Phi^{A_2}$  (as defined by (3)) satisfies  $\Phi^{A_2}(x) \leq C$  on  $B_1$ , for some constant  $C$ . If  $\mathcal{D}(B_1)$  is empty, then  $\Phi$  is a coboundary by Lemma 2.6. Let  $u$  be a limit direction for  $\Phi^{B_1}$ .

The cocycle  $(\Phi_n^{A_2}(x))$ , for a piece of orbit starting and ending in  $A_2$  and for a special sequence of times, can be written as (1) + (2) + (3) where

$$\begin{aligned} (1) &= \sum_{t=0}^{R_{B_1}(x)-1} \Phi(T^t x), \\ (2) &= \Phi_{n_k(x_1)}^{B_1}(x_1), \text{ with } x_1 = T^{R_{B_1}(x)} x, \\ (3) &= \Phi^{A_2}(x_2), \text{ with } x_2 = T_{B_1}^{n_k(x_1)} x_1. \end{aligned}$$

The first term (1) corresponds to the path from  $A_2$  to  $B_1$ . The second term (2) corresponds to visits of the cocycle induced on  $B_1$  to a neighborhood of  $u$  (after normalization) with an arbitrary large norm (such visits exist because  $u$  is a limit direction for the induced cocycle on  $B_1$ ), and the third (3) to the path from  $B_1$  to  $A_2$  with a bounded value of the cocycle by construction.

If we iterate for a long time the induced cocycle (2), the first term (which is fixed) and the third (which is bounded) are small compared with the norm of (2). Then (1) + (2) + (3) gives a value of the induced cocycle on  $A_2$  which satisfy the condition that  $u$  is a limit direction for  $\Phi^{A_2}$ .  $\square$

## 2.2. Essential limit directions.

The observation that the set of limit directions  $\mathcal{D}(\Phi)$  is not a "cohomological invariant" motivates the following definition.

**Definition 2.9.** A direction  $u \in \mathbb{S}^{d-1}$  is called an *essential limit direction* for  $\Phi$ , if, for every subset  $B$  of positive measure,  $u$  is a limit direction for  $\Phi^B$ . The set of essential limit directions is denoted by  $\mathcal{ED}(\Phi)$ .

The set  $\mathcal{ED}(\Phi)$  can be seen as a "boundary" for  $(\Phi_n)$ . It is invariant by cohomology: if  $\Phi_1$  and  $\Phi_2$  are cohomologous, then  $\mathcal{ED}(\Phi_1) = \mathcal{ED}(\Phi_2)$ .

**Theorem 2.10.** *1)  $\mathcal{ED}(\Phi)$  is a closed subset of  $\mathbb{S}^{d-1}$  which is empty if and only if  $\Phi$  is a coboundary. For every  $B$  of positive measure,  $\mathcal{ED}(\Phi^B) = \mathcal{ED}(\Phi)$ .*



2) If  $(\Phi_n)$  is transient and  $\Phi$  is bounded, then  $\mathcal{D}(\Phi)$  is a closed connected non empty subset of  $\mathbb{S}^{d-1}$ .

*Proof.* 1) We have  $\mathcal{ED}(\Phi) = \bigcap \mathcal{D}(\Phi^B)$ , where the intersection is over the family of all measurable subsets of positive measure.

Clearly,  $\mathcal{ED}(\Phi) \subset \mathcal{ED}(\Phi^B)$ . Let  $A$  with  $\mu(A) > 0$ . By Lemma 2.8, there is  $B_1 \subset B$  such that every limit direction for  $\Phi^{B_1}$  is a limit direction for  $\Phi^A$ . If  $u$  is in  $\mathcal{ED}(\Phi^B)$ , then  $u$  is in  $\mathcal{D}(\Phi^{B_1})$ , hence in  $\mathcal{D}(\Phi^A)$ . Therefore  $\mathcal{ED}(\Phi^B) \subset \mathcal{D}(\Phi^A)$ , for all  $A$ , which implies  $\mathcal{ED}(\Phi^B) \subset \mathcal{ED}(\Phi)$ .

2) Let  $u_1$  and  $u_2$  be two accumulation points of  $\Phi_n(x)/\|\Phi_n(x)\|$  and  $\varepsilon > 0$ . For a.e.  $x$ , by transience, for  $N$  big enough, we have  $\|\Phi_n(x)\| \geq \varepsilon^{-1}\|\Phi\|_\infty, \forall n \geq N$ . By definition, there exist  $n > m > N$  such that  $d(\Phi_m(x)/\|\Phi_m(x)\|, u_1) < \varepsilon$  and  $d(\Phi_n(x)/\|\Phi_n(x)\|, u_2) < \varepsilon$ . Moreover, for every  $k$  between  $m$  and  $n - 1$ , one has

$$\begin{aligned} & \left\| \frac{\Phi_k(x)}{\|\Phi_k(x)\|} - \frac{\Phi_{k+1}(x)}{\|\Phi_{k+1}(x)\|} \right\| \\ & \leq \frac{\|\Phi \circ T^k(x)\|}{\|\Phi_{k+1}(x)\|} + \|\Phi_k(x)\| \left| \frac{1}{\|\Phi_{k+1}(x)\|} - \frac{1}{\|\Phi_k(x)\|} \right| \leq 2\varepsilon. \end{aligned}$$

Thus, for every  $\varepsilon > 0$ , one has a finite set  $F_\varepsilon$  of points  $\Phi_k(x)/\|\Phi_k(x)\|$  on the unit sphere that can be used to go from  $u_1$  to  $u_2$  with jumps of length smaller than  $2\varepsilon$ . Now let  $\varepsilon$  tend to zero and consider  $F_\infty$  an accumulation point of  $(F_\varepsilon)_{\varepsilon>0}$  in the set of compact sets of the sphere equipped with the Hausdorff metric. The set  $F_\infty$  is a connected compact set containing  $u_1$  and  $u_2$ .  $\square$

Let  $\tilde{\mathcal{E}}(\Phi)$  be the smallest vector space of  $\mathbb{R}^d$  containing  $\mathcal{E}(\Phi)$ . Using Definition 2.2, we have:

**Theorem 2.11.** *For every non coboundary  $\Phi$ ,  $\mathcal{ED}(\Phi)$  contains  $\mathbb{S}(\tilde{\mathcal{E}}(\Phi))$ , the sphere at infinity of  $\tilde{\mathcal{E}}(\Phi)$ , and is equal to  $\mathbb{S}(\tilde{\mathcal{E}}(\Phi))$  if  $\Phi$  is a regular cocycle.*

**Remarks and questions.** a) A general question is to find the set of limit directions and the set of essential limit directions of a given cocycle. What are the possible shapes of these sets ?

b) The rate of growth of the cocycle plays no role in the "directional process" associated to a cocycle as defined above. This rate could be taken into account by introducing a scaling in the notion of limit directions.

c) Let us call "irreducible" a  $\mathbb{R}^d$ -cocycle which is not cohomologous to a cocycle with values in a vector subspace of dimension  $< d$ . For an irreducible cocycle  $\Phi$  what kind of set  $\mathcal{D}(\Phi)$  can be ? In particular does there exist a recurrent cocycle  $(\Phi_n)$  such that  $\mathcal{D}(\Phi)$  reduces to two antipodal points.

This question is related to the following remark. Let  $\Phi$  and  $\Psi$  with values in  $\mathbb{R}^d$  be given. We say that *the cocycle  $(\Phi_n)$  dominates  $(\Psi_n)$* , if there are  $C$  and  $K$  such that  $\|\Psi_n(x)\| \leq C\|\Phi_n(x)\| + K, \forall n$ .

Clearly this is the case when  $\Psi$  is cohomologous to a multiple of  $\Phi$  with a bounded transfer function. The proposition below is a partial converse.

**Proposition 2.12.** *Assume that  $T_\Phi$  is ergodic on  $X \times \mathbb{R}^d$ . If  $(\Phi_n)$  dominates  $(\Psi_n)$ , then  $\Psi$  is cohomologous to  $c\Phi$  for a constant  $c$ .*

*Proof.* Let  $I$  be a compact neighborhood of  $\{0\}$ . Let  $y$  be in  $I$ . For the times  $n_k(x, y)$  such that  $y + \Phi_{n_k(x, y)}(x) \in I$ ,  $|\Psi_n(x)|$  is bounded. The cocycle  $\Psi_n^{Z_I}(x)$  induced of  $\Psi$  on the set  $Z_I := X \times I$  is bounded. Therefore the function  $F$  defined on  $X \times \mathbb{R}$  by  $F(x, y) = \Psi(x)$  is a coboundary for the map  $T_\Phi$ : there is  $H(x, y)$  such that  $F(x, y) = \Psi(x) = H(Tx, y + \Phi(x)) - H(x, y)$ .

For every  $a \in \mathbb{R}^d$ , the function  $(x, y) \rightarrow H(x, y + a) - H(x, y)$  is  $T_\Phi$ -invariant, hence a.e. constant by ergodicity of  $T_\Phi$ : for a.e.  $(x, y)$ , there is  $c(a)$  such that  $H(x, y + a) = c(a) + H(x, y)$ . By the theorem of Fubini, for a.e.  $y$ ,  $H(x, y + a) = c(a) + H(x, y)$ , for almost every  $(x, a)$ , and  $a \rightarrow c(a)$  is Lebesgue measurable. Let us take  $y_0$  satisfying this property. We have  $H(x, a + y_0) = c(a) + H(x, y_0)$ , for a.e.  $(x, a)$ ; hence, with  $u(a) = c(a - y_0)$  and  $h(x) = H(x, y_0)$ :

$$H(x, a) = u(a) + h(x), \text{ for a.e. } (x, a).$$

The relation  $H(x, y + a) = c(a) + H(x, y)$  reads:  $u(y + a) + h(x) = c(a) + u(y) + h(x)$  which shows that  $u$  is an additive function.

Therefore  $H(x, y) = cy + h(x)$  for a constant  $c$  and a measurable function  $h$  on  $X$  and we have  $\Psi(x) = c\Phi(x) + h(Tx) - h(x)$ .  $\square$

### 2.3. A $G_\delta$ -property.

Suppose that the map  $T = T(\theta)$  and the function defining the cocycle  $\Phi = \Phi^\theta$  depend on a parameter  $\theta$ . Suppose that  $\Theta$ , the set of parameters, is a metric space and that the dependence of  $T(\theta)$  and  $\Phi^\theta$  is piecewise continuous. We denote by  $\mathcal{V} = \mathcal{V}(u)$  a countable basis of open neighborhoods of a direction  $u$  in  $\mathbb{S}^{d-1}$ .

**Theorem 2.13.** *Suppose that in the set of parameters  $\Theta$  there is a dense set  $\mathcal{T}$  of values such that the corresponding set of limit directions is  $\mathbb{S}^{d-1}$ . Then there is a dense  $G_\delta$ -set in  $\Theta$  with the same property.*

*Proof.* We can assume that  $\mathcal{T}$  is countable:  $\mathcal{T} = \{\theta_i, i = 1, 2, \dots\}$ . Fix a direction  $u \in \mathbb{S}^{d-1}$ . For  $\theta \in \mathcal{T}$ , for a.e.  $x \in X$ ,  $u$  is a limit direction for  $(\Phi_n^\theta(x))$ . Let  $K$  be a compact set of positive measure in  $X$  such that for every  $i$ ,

$$K \subset \{x \in X : u \text{ is a limit direction for } \Phi_n^{\theta_i}(x)\}.$$

For a fixed  $x$ , for  $M \geq 1$  and  $V \in \mathcal{V}(u)$ , the set

$$\tilde{B}_n^{x, V, M} = \{\theta : \|\Phi_n^\theta(x)\| > M \text{ and } \Phi_n^\theta(x)/\|\Phi_n^\theta(x)\| \in V\}$$

is an open set.

If  $W$  is an open set in  $X$ , let

$$\tilde{B}_n^{W, V, M} := \{\theta : \|\Phi_n^\theta(y)\| > M \text{ and } \Phi_n^\theta(y)/\|\Phi_n^\theta(y)\| \in V, \forall y \in W\}.$$

Let  $V \in \mathcal{V}(u)$ ,  $M \in \mathbb{N}$  and  $\theta_i \in \mathcal{T}$ . For each  $x \in K$ , there exists  $n$  such that  $\theta_i \in \tilde{B}_n^{x,V,M}$ . By continuity  $\theta_i \in \tilde{B}_n^{y,V,M}$  for  $y$  in an open neighborhood of  $x$ . Thus there are finitely many open sets  $W_{(V,M)}^1, \dots, W_{(V,M)}^{r_i(V,M)}$  covering  $K$  and integers  $n_{(V,M)}^1, \dots, n_{(V,M)}^{r_i(V,M)}$  such that

$$\theta_i \in \bigcap_{j=1, \dots, r_i(V,M)} \tilde{B}_{n_{(V,M)}^j}^{W_{(V,M)}^j, V, M}.$$

This proves that, for every  $y \in K$ , there are  $j \in \{1, \dots, r_i(V, M)\}$  and  $n_j$  such that

$$\|\Phi_{n_j}^{\theta_i}(y)\| > M \text{ and } \Phi_{n_j}^{\theta_i}(y)/\|\Phi_{n_j}^{\theta_i}(y)\| \in V.$$

For every  $i$ ,  $\theta_i$  belongs to the open set  $\bigcup_i \bigcap_{j \in \{1, \dots, r_i\}} \tilde{B}_{n_{(V,M)}^j}^{W_{(V,M)}^j, V, M}$ . The dense set of parameters  $\mathcal{T} = \{\theta_i\}$  is contained in the countable intersection of open sets:

$$(10) \quad \bigcap_{V \in \mathcal{V}(u), M \geq 1} \bigcup_i \bigcap_{j \in \{1, \dots, r_i(V, M)\}} \tilde{B}_{n_{(V,M)}^j}^{W_{(V,M)}^j, V, M}.$$

Now, suppose that the parameter belongs to the dense  $G_\delta$ -set defined above by (10).

For every  $V \in \mathcal{V}(u)$  and  $M \in \mathbb{N}$ , there is  $i$  such that

$$\theta \in \bigcap_{j \in \{1, \dots, r_i(V, M)\}} \tilde{B}_{n_{(V,M)}^j}^{W_{(V,M)}^j, V, M},$$

i.e., for every  $V \in \mathcal{V}(u)$  and  $M \in \mathbb{N}$ , there are  $i$  and  $j \in \{1, \dots, r_i(V, M)\}$  such that

$$\forall y \in W_{(V,M)}^j, \|\Phi_{n_{(V,M)}^j}^\theta(y)\| > M \text{ and } \Phi_{n_{(V,M)}^j}^\theta(y)/\|\Phi_{n_{(V,M)}^j}^\theta(y)\| \in V.$$

As  $W_{(V,M)}^j, j = 1, \dots, r_i(V, M)$ , is a covering of  $K$ , for each point  $y \in K$ , for all  $V \in \mathcal{V}(u)$ , all  $M \in \mathbb{N}$ , there is  $n$  such that  $\|\Phi_n^\theta(y)\| > M$  and  $\Phi_n^\theta(y)/\|\Phi_n^\theta(y)\| \in V$ .

Therefore for each  $y \in K$ ,  $u$  is a limit direction. As it is a 0-1-property, the property that  $u$  is a limit direction holds for a.e.  $x$ .  $\square$

#### 2.4. Limit directions and limit distributions.

**Lemma 2.14.** *Suppose that, for a sequence of integers  $(k_n)$  and a sequence  $(a_n)$  tending to  $\infty$ ,  $(\Phi_n)$  satisfies a limit theorem in distribution:*

$$a_n^{-1} \Phi_{k_n} \xrightarrow{\text{distrib}} \mathcal{L},$$

where  $\mathcal{L}$  is a probability measure on  $\mathbb{R}^d$  giving a positive probability to each non empty open set. Then the set  $\mathcal{D}(\Phi)$  of limit directions of  $(\Phi_n)$  is  $\mathbb{S}^{d-1}$ . This applies in particular if  $(\Phi_n)$  satisfies a non degenerated CLT for a subsequence and an adapted normalization.

*Proof.* Suppose that  $u \in \mathbb{S}^{d-1}$  is not a limit direction for  $(\Phi_n)$ . By the dichotomy (cf. Remark 1 a)), there is  $M$  and an open regular neighborhood  $V = V(u)$  of  $u$  in  $\mathbb{R}^d$  such that, a.e.  $x$  belongs for some  $N$  to the set

$$C_N := \{x : \forall n \geq N, \|\Phi_{k_n}(x)\| \leq M \text{ or } \Phi_{k_n}(x)/\|\Phi_{k_n}(x)\| \notin V(u)\}.$$

The sequence of sets  $(C_N)$  is increasing and  $\mu(\bigcup_N C_N) = 1$ .

From the assumption, we have

$$\liminf_{n \rightarrow +\infty} \mu\{x \in X : a_n^{-1}\Phi_{k_n}(x) \in V\} \geq \mathcal{L}(V) > 0.$$

Therefore for any  $\alpha > 0$ , there is  $N$  such that, for  $n \geq N$ , there is a set  $B$  in  $X$  of measure  $> \frac{\mathcal{L}(V)}{2}$  such that:  $\|\Phi_{k_n}(x) - a_n u\| \leq \alpha a_n$ , which implies

$$(1 - \alpha)a_n \leq \|\Phi_{k_n}(x)\| \leq (1 + \alpha)a_n, \text{ for } x \in B.$$

Hence:  $\|\Phi_{k_n}(x)/\|\Phi_{k_n}(x)\| - u\| \leq \frac{\alpha}{1-\alpha}$  on a set of measure  $\frac{\mathcal{L}(V)}{2} > 0$ . If we take  $N$  such that  $\mu(C_N) > 1 - \frac{\mathcal{L}(V)}{2}$ , there is a contradiction for  $n > N$  big enough.

This applies when  $\mathcal{L} = \mathcal{N}(0, \Gamma)$  where  $\Gamma$  is a non degenerated covariance matrix.  $\square$

## 2.5. Oscillations of 1-dimensional cocycles.

We discuss now the notion of limit directions in the special case of cocycles with values in  $\mathbb{R}$ . About oscillations of 1-dimensional cocycles, let us mention the work of Derriennic ([De10]) where other references on the subject, in particular of Tanny, Kesten, Wos, can also be found. For completeness we give below a short presentation, related to the notion of limit direction, of some results on 1-dimensional cocycles.

We consider a conservative transformation  $T$  of a space  $(X, \mu)$  where  $\mu$  is  $\sigma$ -finite and non singular for  $T$ . Notice that in Lemmas 2.15 and 2.16 below, ergodicity is not assumed. Recall that equalities between functions are understood  $\mu$ -a.e.

In this subsection, we denote by  $\varphi$  and  $(\varphi_n)$  respectively a given measurable function and the corresponding cocycle over  $(X, \mu, T)$ . We define two sets, clearly  $T$ -invariant (cf. (5)):

$$(11) \quad F_\varphi^+ := (\inf_n \varphi_n(\cdot) > -\infty), \quad F_\varphi^- := (\sup_n \varphi_n(\cdot) < +\infty).$$

Let us recall the following classical lemma (filling scheme).

**Lemma 2.15.** *There are two functions  $h^+$  and  $g^+$  defined on  $F_\varphi^+$  (resp.  $h^-$  and  $g^-$  defined on  $F_\varphi^-$ ) with values in  $[0, +\infty[$  such that*

$$(12) \quad \varphi(x) = h^+(Tx) - h^+(x) + g^+(x), \text{ for } \mu\text{-a.e. } x \in F_\varphi^+,$$

$$(13) \quad \varphi(x) = -h^-(Tx) + h^-(x) - g^+(x), \text{ for } \mu\text{-a.e. } x \in F_\varphi^-,$$

On the invariant set  $F_\varphi^{+\infty} := (\varphi_n(\cdot) \rightarrow +\infty)$ , we have  $\sum_{k=0}^{\infty} g^+(T^k x) = +\infty$ .

If  $(\varphi_n)$  is recurrent, then the space  $X$  decomposes in two invariant sets, each of them possibly of zero measure, one on which  $\varphi$  is a coboundary, the other on which  $\sup_n \varphi_n(\cdot) = +\infty$  and  $\inf_n \varphi_n(\cdot) = -\infty$ .

*Proof.* Let  $m_n(x) := \min_{1 \leq k \leq n} (\varphi_k(x))$ ,  $n \geq 1$ . We have

$$m_{n+1}(x) = \min(\varphi(x), \varphi(x) + m_n(Tx)) = \begin{cases} \varphi(x) - m_n^-(Tx), & \text{if } m_n(Tx) \leq 0, \\ \varphi(x) = \varphi(x) - m_n^-(Tx), & \text{if } m_n(Tx) > 0, \end{cases}$$

which implies  $m_{n+1}(x) = \varphi(x) - m_n^-(Tx)$ . Since the limit  $m_\infty(x) := \lim_n m_n(x)$  is finite on  $F_\varphi^+$ , it follows:

$$\varphi(x) = m_\infty^-(Tx) - m_\infty^-(x) + m_\infty^+(x), \quad x \in F_\varphi^+.$$

This gives the decomposition (12) on  $F_\varphi^+$ , with  $h^+ = m_\infty^-$  and  $g^+ = m_\infty^+$ . We get (13) by changing  $\varphi$  into  $-\varphi$ .

On the invariant set  $F_\varphi^{+\infty} = (\varphi_n(\cdot) \rightarrow +\infty)$  the decomposition (12) holds. Let us show that  $\sum_{k=0}^\infty g^+(T^k x) = +\infty$ .

We have  $\varphi_n(x) = h^+(T^n x) - h^+(x) + \sum_{k=0}^{n-1} g^+(T^k x)$ . Let  $M_K$  be the subset of  $F_\varphi^{+\infty}$  where  $h^+$  is bounded by a finite constant  $K$ . By conservativity of  $T$ , for a.e.  $x$  in  $M_K$  there is a subsequence  $(n_j(x))$  such that  $T^{n_j(x)}(x) \in M_K$  and therefore  $\sum_{k=0}^{n_j(x)-1} g^+(T^k x) \rightarrow +\infty$ . It follows  $\sum_{k=0}^\infty g^+(T^k x) = +\infty$  for a.e.  $x$  in  $M_K$  and, since  $K$  is arbitrary,  $\sum_{k=0}^\infty g^+(T^k x) = +\infty$  a.e.

Suppose now that  $(\varphi_n)$  is recurrent. Let  $g_\infty^+(x) = \sum_{k=0}^\infty g^+(T^k x) \in [0, +\infty]$ . The induced cocycle  $(\varphi_n^B)$  is recurrent for any set  $B$  on which  $h^+$  is bounded and therefore  $g_\infty^+(x) < +\infty$ , a.e. on  $B$ . As the sets  $B$  cover  $F_\varphi^+$ , this implies  $g_\infty^+(x) < +\infty$ , a.e. on  $F_\varphi^+$ . Therefore  $g^+(x) = g_\infty^+(x) - g_\infty^+(Tx)$ , and the restriction of  $\varphi$  to the invariant set  $F_\varphi^+$  is a coboundary. Likewise,  $\varphi$  is a coboundary on the invariant set  $F_\varphi^-$ .

So we have proved that, for any recurrent cocycle, the space  $X$  decomposes in two sets, the set  $F_\varphi^+ \cup F_\varphi^-$  on which  $\varphi$  is a coboundary and its complement on which  $\varphi_n$  oscillates between  $+\infty$  and  $-\infty$ .  $\square$

The lemma implies that  $\varphi$  is a coboundary on the invariant set  $\{x : \varphi_n(x) \text{ is bounded}\}$ . It is well known that, if  $(\varphi_n)$  is uniformly bounded, then the transfer function is bounded.

The previous lemma gives a simple way to prove and to slightly extend a result of Kesten on the rate of divergence in dimension 1 of a non recurrent cocycle. We consider a conservative dynamical system with a  $\sigma$ -finite invariant measure  $\mu$ . The  $\sigma$ -algebra of  $T$ -invariant sets is denoted by  $\mathcal{I}$ .

As  $\mu$  is  $\sigma$ -finite, we can choose a function  $p$  on  $X$  such that  $\mu(p) = 1$  and  $0 < p(x) \leq 1, \forall x$ . By the ratio ergodic theorem  $\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} f(T^k x)}{\sum_{k=0}^{n-1} p(T^k x)} = \mathbb{E}_{p\mu}[\frac{f}{p} | \mathcal{I}](x)$ ,  $\mu$ -a.e.  $x \in X$ , for  $f \in \mathbb{L}^1(\mu)$ . We have  $p_n(x) = \sum_{k=0}^{n-1} p(T^k x) \rightarrow \infty$  since the system is conservative.

**Lemma 2.16.** (cf. also [Ke75], [De10]) Suppose that the  $T$ -invariant measure  $\mu$  is conservative  $\sigma$ -finite.

1) If  $\varphi$  is a nonnegative measurable function, then for  $\mu$ -a.e.  $x$  the sum  $\sum_0^\infty \varphi(T^k x)$  is either 0 or  $+\infty$ , and  $\liminf_n \frac{\varphi_n(x)}{p_n(x)} \in ]0, +\infty]$  on the set  $\{\sum_0^\infty \varphi(T^k \cdot) \neq 0\}$ .

2) For any measurable  $\varphi$ ,  $\liminf_n \frac{\varphi_n(x)}{p_n(x)} > 0$  for  $\mu$ -a.e.  $x$  on the set  $(\varphi_n(\cdot) \rightarrow +\infty)$ .

The cocycle  $(\varphi_n)$  is recurrent on the invariant set  $\{x : \lim \frac{\varphi_n(x)}{p_n(x)} = 0\}$ .

3) If  $\mu$  is a  $T$ -invariant probability measure, then for any measurable  $\varphi$ ,  $\liminf_n \frac{1}{n} \varphi_n(x) > 0$  for  $\mu$ -a.e.  $x$ , on the set  $(\varphi_n(\cdot) \rightarrow +\infty)$  and the cocycle  $(\varphi_n)$  is recurrent on the invariant set  $\{x : \lim \frac{1}{n} \varphi_n(x) = 0\}$ .

*Proof.* 1a) First, suppose that  $\varphi = 1_B$  where  $B$  is a measurable subset. For  $N \geq 0$ , let  $B^{(N)} := \cup_0^N T^{-k} B$ ,  $B^{(\infty)} = \cup_0^\infty T^{-k} B$ . We have  $T^{-1} B^{(\infty)} \subset B^{(\infty)}$ , hence  $T^{-1} B^{(\infty)} = B^{(\infty)}$  up to a negligible set as  $(X, \mu, T)$  is conservative.

On the complement of  $B^{(\infty)}$ , we have  $\sum_0^n 1_B(T^k x) = 0$ . By the ratio ergodic theorem  $\lim_n \frac{1}{p_n(x)} \sum_0^{n-1} 1_B(T^k x) p(T^k x) = \mathbb{E}_{p\mu}(1_B | \mathcal{I})(x)$ .

As  $\sum_{j=0}^{L-1} 1_B(T^j x) \geq 1_{B^{(L)}}(x)$  and  $\lim_n p_n(x) = +\infty$ , we have

$$\begin{aligned} \liminf_n \frac{1}{p_n(x)} \sum_{k=0}^{n-1} 1_B(T^k x) &\geq \frac{1}{L} \liminf_n \frac{1}{p_n(x)} \sum_{k=0}^{n-1} 1_{B^{(L)}}(T^k x) \\ &\geq \frac{1}{L} \lim_n \frac{1}{p_n(x)} \sum_{k=0}^{n-1} 1_{B^{(L)}}(T^k x) p(T^k x) = \frac{1}{L} \mathbb{E}_{p\mu}(1_{B^{(L)}} | \mathcal{I})(x). \end{aligned}$$

Therefore from the relation

$$\bigcup_L \uparrow \{x : \mathbb{E}_{p\mu}(1_{B^{(L)}} | \mathcal{I})(x) > 0\} = \{x : \mathbb{E}_{p\mu}(1_{B^{(\infty)}} | \mathcal{I})(x) > 0\} = 1_{B^{(\infty)}},$$

it follows

$$(14) \quad \liminf_n \frac{1}{p_n(x)} \sum_{k=0}^{n-1} 1_B(T^k x) > 0 \text{ on } B^{(\infty)}.$$

1b) Now let  $\varphi$  be any nonnegative function. For  $j \in \mathbb{Z}$ , let  $B_j := \{2^j \leq \varphi < 2^{j+1}\}$ . We get:  $\sum_0^\infty \varphi(T^k x) > 0 \Leftrightarrow \exists j : x \in B_j^\infty$ . Using (14) applied to the sets  $B_j$  and the inequality  $\varphi \geq \sum_{j=-\infty}^{+\infty} 2^j 1_{B_j}$ , we obtain:

$$\liminf_n \frac{1}{p_n(x)} \sum_0^{n-1} \varphi(T^k x) \geq \sum_{j=-\infty}^{+\infty} 2^j \liminf_n \frac{1}{p_n(x)} \sum_0^{n-1} 1_{B_j}(T^k x) > 0, \text{ on } \left(\sum_0^\infty \varphi(T^k \cdot) > 0\right).$$

2) As (12) of Lemma 2.15 holds on the set  $F_\varphi^{+\infty} = (\varphi_n(\cdot) \rightarrow +\infty)$  and  $\sum_{k=0}^\infty g^+(T^k x) = +\infty$ , we can apply 1) to  $g^+$ . Since  $\varphi_n(x) \geq -h^+(x) + g_n^+(x)$ , we get:

$$\liminf_n \frac{\varphi_n(x)}{p_n(x)} \geq \liminf_n \frac{g_n^+(x)}{p_n(x)} > 0, \text{ for } \mu\text{-a.e. } x \in F_\varphi^{+\infty}.$$

This implies that  $(\varphi_n(\cdot))$  is recurrent on the invariant set  $\{x : \lim_{p_n(x)} \frac{\varphi_n(x)}{p_n(x)} = 0\}$ , since we can not have  $\varphi_n(x) \rightarrow +\infty$  or  $\varphi_n(x) \rightarrow -\infty$  on this set.

In particular, if  $T$  is ergodic and  $\varphi$  integrable with  $\mu(\varphi) = 0$ , we have  $\lim_n \frac{\varphi_n(x)}{p_n(x)} = \mu(\varphi) = 0$  which implies the recurrence of the cocycle  $(\varphi_n)$ .

3) When the measure is finite, then we take  $p(x) = 1$  and  $p_n(x)$  is replaced by  $n$ .  $\square$

**Proposition 2.17.** *For a 1-dimensional cocycle  $(\varphi_n)$  generated by  $\varphi$  over an ergodic dynamical system, if  $\varphi$  is not a coboundary, one of the following (exclusive) properties is satisfied:*

- 1)  $\limsup_n \varphi_n = -\liminf \varphi_n = +\infty$ ,
- 2)  $\varphi_n$  tends to  $+\infty$ ,
- 3)  $\varphi_n$  tends to  $-\infty$ .

*Proof.* By ergodicity, with the notation (11), one of the sets  $(F_\varphi^+ \cup F_\varphi^-)^c$ ,  $F_\varphi^+$ ,  $F_\varphi^-$  has full measure.

The first case is equivalent to property 1) and to the equality  $\mathcal{ED}(\varphi) = \{-\infty, +\infty\}$ . Suppose now that  $F_\varphi^+$  has full measure. Then we have the decomposition (12), Lemma 2.15, with equality a.e. Hence  $\varphi_n(x) \geq -h^+(x) + g_n^+(x)$ , for  $\mu$ -a.e.  $x$ . Since  $\varphi$  is not a coboundary,  $g^+$  is non negligible. This implies property 2). Likewise 3) holds if  $F_\varphi^-$  has full measure.  $\square$

This leads to the following remarks.

- if the cocycle  $(\varphi_n)$  is recurrent, it oscillates between  $+\infty$  and  $-\infty$ , unless  $\varphi$  is a coboundary with a transfer function bounded from above or below;
- if  $\varphi$  is a coboundary,  $\varphi = T\psi - \psi$ , then if  $\psi$  is not essentially bounded from above (resp. from below), then  $\limsup_n \varphi_n = +\infty$  (resp.  $\liminf \varphi_n = -\infty$ );
- we have  $-\infty \notin \mathcal{D}(\varphi)$  if and only if  $\varphi = Th - h + g$ , with  $h, g$  non negative and  $g$  non negligible. This is equivalent to  $\lim_n \varphi_n = +\infty$ ;
- $\mathcal{D}(\varphi)$  is empty if  $\varphi$  is a coboundary,  $\varphi = Th - h$ , with  $h$  essentially bounded.

For the constructions below, we need the following lemma:

**Lemma 2.18.** *Let  $(\ell_n)$  be a strictly increasing sequence of integers. For any ergodic dynamical system there is  $h$  non negative such that, for a.e.  $x$ ,  $h(T^n x) \geq \ell_n$  infinitely often.*

*Proof.* There exists a strictly increasing sequence of positive real numbers  $(c_j)$  and a strictly increasing sequence of natural numbers  $(n_j)$ , both tending to infinity, and a non negative measurable function  $f$  such that

$$\lim_j \frac{1}{c_j n_j} \sum_{k=1}^{n_j} f(T^k x) = +\infty, \text{ a.e.}$$

We put  $d_k = c_j$ , for  $n_{j-1} \leq k < n_j$ . For a.e.  $x$ , for  $j$  big enough, we can define a non decreasing sequence  $(k_j(x))$  such that  $\lim_j k_j(x) = +\infty$  and  $f(T^{k_j(x)} x) \geq c_j \geq d_{k_j(x)}$ . (Put  $k_j(x) := \max\{k \leq n_j : f(T^k x) \geq c_j\}$ .)

Now we can define a non decreasing function  $\gamma$  on  $\mathbb{R}^+$  by putting  $\gamma(y) = \ell_k$ , for  $d_k \leq y < d_{k+1}$ . In particular, we have  $\gamma(d_k) = \ell_k$ .

Let  $h(x) := \gamma(f(x))$ . Then we have:  $h(T^k x) = \gamma(f(T^k x)) \geq \gamma(d_k) = \ell_k$ , if  $f(T^k x) \geq d_k$ .

Therefore for a.e.  $x$ , for  $j$  big enough,  $h(T^{k_j(x)}(x)) \geq \ell_{k_j(x)}$ .  $\square$

For every  $B \subset X$  of positive measure,  $\varphi$  is a coboundary, if and only if the induced cocycle  $\varphi^B$  is a coboundary for the induced map  $T_B$ . If  $\varphi$  is not a coboundary, then the inclusion  $\mathcal{D}(\varphi^B) \subset \mathcal{D}(\varphi)$  is general, but can be strict.

*Example:* Let  $B$  with  $\mu(B) > 0$ . Take  $\varphi = Th - h + 1$ , with  $h(x) \geq 0$  on  $B$ . Then, if  $R_n(x)$  denotes the  $n$ -th return time to  $B$ , we have

$$\varphi_n^B(x) = h(T_B^n x) - h(x) + R_n(x) \geq -h(x) + R_n(x) \rightarrow +\infty.$$

We can choose  $h$  on  $B^c$  such that, for the cocycle  $\varphi_n(x) = h(T^n x) - h(x) + n$ , we have  $h(T^n x) \leq -n^2$  infinitely often (Lemma 2.18). Therefore  $-\infty \in \mathcal{D}(\varphi) \neq \mathcal{D}(\varphi_B) = \{+\infty\}$ .

### Reverse cocycle

Recall that the reverse cocycle  $(\check{\varphi}_n)_{n \geq 0}$  is defined by  $\check{\varphi}_0 = 0$  and

$$\check{\varphi}_n(x) = -\varphi_n(T^{-n}x) = -\sum_{k=1}^n \varphi(T^{-k}x), \quad \text{for } n \geq 1.$$

For an ergodic system, if  $\varphi$  is integrable and if  $\lim_n \varphi_n = +\infty$ , then  $\lim_n \check{\varphi}_n = -\infty$ , since both conditions are equivalent to  $\mu(\varphi) > 0$ .

If  $\varphi$  is non integrable, we can have  $\lim_n \varphi_n = +\infty$  and  $\limsup_n \check{\varphi}_n = +\infty$ . (see also [De10]).

*Example:* Let  $\varphi = Th - h + 1$ , with  $h$  non negative. We have  $\lim_n \varphi_n(x) = +\infty$ . The reverse cocycle reads

$$\check{\varphi}_n(x) = -\varphi_n(T^{-n}x) = -h(x) + h(T^{-n}x) - n.$$

If  $h$  is chosen such that the inequality  $h(T^{-n}x) \geq n^2$  occurs infinitely often for a.e.  $x$  (Lemma 2.18), then  $+\infty$  is a limit direction for the reverse cocycle.

## A result in dimension 2

Let us mention a partial result for 2-dimensional cocycles:

**Proposition 2.19.** *Let  $\Phi : X \rightarrow \mathbb{R}^2$  be an integrable and centered function. If  $(\Phi_n)$  is a transient cocycle over an ergodic dynamical system, then  $\mathcal{D}(\Phi) \cup (-\mathcal{D}(\Phi)) = \mathbb{S}^1$  for a.e.  $x \in X$ .*

*Proof.* We denote by  $\langle u, v \rangle$  the scalar product in  $\mathbb{R}^2$ . Let  $v \in \mathbb{S}^1$ , and let  $v^\perp \in \mathbb{S}^1$  be such that  $\langle v, v^\perp \rangle = 0$ . The function  $x \mapsto \langle \Phi(x), v^\perp \rangle$  has zero integral and  $T$  is ergodic, hence the cocycle  $\langle (\Phi_n), v^\perp \rangle$  is recurrent. For a.e.  $x \in X$  there is a



sequence  $n_k(x) \rightarrow \infty$  and  $c > 0$  such that  $|\langle \Phi_{n_k}(x), v^\perp \rangle| < c$ . As  $(\Phi_n)$  is transient, for a.e.  $x$   $|\langle \Phi_{n_k}(x), v \rangle|$  is not bounded. There is a subsequence  $(n_{k_j})$  such that  $\Phi_{n_{k_j}}(x)/\|\Phi_{n_{k_j}}(x)\|$  converges to  $v$  or  $-v$ , i.e.,  $v$  or  $-v \in \mathcal{D}(\Phi)(x)$ .  $\square$

By what precedes and Theorem 2.10, when  $\Phi$  is bounded, the set  $\mathcal{D}(\Phi)$  is an arc of a circle with length  $\geq \frac{1}{2}$ .

### 3. Application of the CLT, martingales, invariance principle

For a large class of dynamical systems of hyperbolic type, the method introduced by M. Gordin in 1969 gives a way to reduce, up to a regular coboundary, a Hölderian function  $\Phi$  to a function satisfying a martingale condition. This allows to prove for regular functions which are not coboundaries, not only a CLT, but also a CLT for subsequences of positive density and the functional CLT (or the invariance principle).

In this subsection, we recall some results for martingale increments and briefly mention their application to find the set of essential directions..

#### 3.1. Martingale methods and essential limit directions.

The theorem of Ibragimov and Billingsley stated in terms of dynamical systems, gives a CLT which can be extended to several improvements:

**Proposition 3.1.** *Let  $(X, \mathcal{A}, \mu, T)$  be an ergodic invertible dynamical system and  $\mathcal{F}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$  such that  $\mathcal{F} \subset T^{-1}\mathcal{F}$ . Let  $\Phi$  be a  $\mathbb{R}^d$ -valued square integrable function,  $\mathcal{F}$ -measurable and such that the sequence  $(\Phi \circ T^n)_{n \in \mathbb{Z}}$  is a sequence of martingale increments with respect to  $(T^{-n}\mathcal{F})$  (equivalently by stationarity:  $\mathbb{E}(\Phi|T\mathcal{F}) = 0$ ).*

*If  $\Phi$  is non contained a.s. in a fixed hyperplane, the cocycle  $(\Phi_n)$  is such that  $(\frac{1}{\sqrt{n}}\Phi_n)_{n \geq 1}$  has asymptotically a Gaussian law, with a non degenerated covariance matrix  $\Gamma$ .*

*For every strictly increasing sequence of measurable functions  $(k_n)_{n \geq 1}$  with values in  $\mathbb{N}$  such that, for a constant  $a \in ]0, \infty[$ ,  $\lim_n \frac{k_n(x)}{n} = a$  exists a.e. we have:*

$$\frac{1}{\sqrt{n}}\Phi_{k_n(\cdot)}(\cdot) \xrightarrow{\mathcal{L}} \mathcal{N}(0, a^{-1}\Gamma).$$

*Moreover the cocycle  $(\Phi_n)$  satisfies the invariance principle.*

**Theorem 3.2.**  $\mathcal{ED}(\Phi) = \mathbb{S}^{d-1}$  under the conditions of the previous proposition.

*Proof.* Let  $B$  be a subset of positive measure and let  $(R_n(x))$  be the sequence of visit times in  $B$ . The induced cocycle  $(\Phi_n^B)$  is obtained by sampling the cocycle  $(\Phi_n)$  at the random times  $R_n$  of visits to  $B$ .

We have (Kac lemma):  $\lim_n \frac{R_n(x)}{n} = \frac{1}{\mu(B)}$ , since by the ergodic theorem:

$$\frac{n}{R_n(x)} = \frac{1}{R_n(x)} \sum_{j=0}^{R_n(x)-1} 1_B(T^j x) \rightarrow \mu(B).$$

Therefore  $(\Phi_n^B)$  satisfies the CLT, with the same covariance matrix as for the cocycle  $(\Phi_n)$  up to a scalar. By Lemma 2.14, this implies the result.  $\square$

If the covariance matrix is degenerated, the set  $\mathcal{ED}(\Phi)$  is the unit sphere of a subgroup isomorphic to  $\mathbb{R}^{d'}$ , for  $d' < d$ .

### *Reduction by cohomology to martingale increments*

When Gordin's method can be applied, using Theorem 3.2 and the fact that the set of essential limit directions is the same for two cohomologous cocycles, we obtain  $\mathcal{ED}(\Phi) = \mathbb{S}^{d'-1}$ , for some  $d' \leq d$ , if  $\Phi$  is Hölderian with values in  $\mathbb{R}^d$ .

This method can be used for Hölderian functions in many systems, among which: piecewise continuous expansive maps of the interval, toral automorphisms, geodesic and diagonal flows on homogeneous spaces of finite volume, dispersive billiards in the plane.

Let us give an explicit example.

**Proposition 3.3.** *Let  $T$  be an ergodic endomorphism of the torus  $\mathbb{T}^r$ ,  $r \geq 1$ , endowed with the Lebesgue measure. If  $\Phi$  is a Hölderian function with values in  $\mathbb{R}^d$ , then  $\mathcal{ED}(\phi) = \mathbb{S}^{d'-1}$ , for  $d' \leq d$ .*

*Proof.* The function  $\Phi$  is cohomologous to  $\Psi$  such that  $(\Psi \circ T^n)$  is a sequence of vector  $d'$ -dimensional martingale increments (with  $d' \leq d$ ) (cf. [LB99]). We can apply Proposition 3.1, then Theorem 3.2.  $\square$

The situation for the models where Gordin's method is available is comparable to that of cocycles which are regular in the sense of Definition 2.2.

In the last section we will deduce a stronger property from the invariance principle.

### **3.2. Invariance principle and behavior of the directional process.**

Let  $(X, \mu, T)$  be an ergodic dynamical system and let  $\Phi$  be a measurable function on  $X$  with values in  $\mathbb{R}^d$ ,  $d \geq 2$ . In this subsection,  $\Phi$  is assumed to be bounded and centered.

We are going to give conditions on  $(\Phi_n)$  which imply the property (9) introduced in the first section for the directional process  $Z_n = \tilde{\Phi}_n(\cdot) = \frac{\Phi_n(\cdot)}{\|\Phi_n(\cdot)\|}$ .

We denote by  $(W_n^\Phi)_{n \geq 1}$  (or simply  $(W_n)_{n \geq 1}$ ) the interpolated piecewise affine process with continuous paths defined for  $x \in X$  and  $n \geq 1$  by

$$W_n^\Phi(x, s) = \Phi_k(x) + (ns - k)\Phi(T^k x) \text{ if } s \in \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

If  $C$  is a cone with non empty interior  $\overset{\circ}{C}$  and boundary  $\partial C$  of measure 0, the amount of time spent by  $W_n^\Phi(x, s)$  in  $C$  is

$$\tau_{n,C}^\Phi(x) = \int_0^1 \mathbf{1}_C(W_n^\Phi(x, s)) ds.$$

Recall that the *invariance principle* for  $\Phi$ , sometimes called Donsker's invariance principle, means here that the process  $(\frac{W_n^\Phi(x, \cdot)}{\sqrt{n}})_{n \geq 1}$  (defined on the probability space  $(X, \mu)$  and with values in the space  $(\mathcal{C}_d([0, 1], \|\cdot\|_\infty)$  of continuous functions from  $[0, 1]$  to  $\mathbb{R}^d$  endowed with the uniform norm) converges in distribution to the standard Brownian motion in  $\mathbb{R}^d$  (cf. [Bi]).

As mentioned before, the invariance principle, a by-product of the martingale method, is valid for large classes of regular functions in many dynamical systems of hyperbolic type.

**Theorem 3.4.** *Suppose that  $(X, \mu, T)$  is ergodic, that the invariance principle is satisfied for a centered bounded function  $\Phi : X \rightarrow \mathbb{R}^d$  and that  $C$  is a cone with non empty interior in  $\mathbb{R}^d$  and with a complement with non empty interior. Then, for almost every  $x$ ,*

$$\limsup_{n \rightarrow \infty} \tau_{n,C}^\Phi(x) = 1 \text{ and } \liminf_{n \rightarrow \infty} \tau_{n,C}^\Phi(x) = 0$$

We need preliminary lemmas before the proof of Theorem 3.4. Firstly, let us remark that the property stated in the theorem holds for the Brownian motion.

### Visit of the Brownian motion in cones

Let  $(B_s)$  denote the standard Brownian motion in  $\mathbb{R}^d$ , for  $d \geq 2$ . Consider a cone  $C$  with non empty interior  $\overset{\circ}{C}$  in  $\mathbb{R}^d$  and with a complement with non empty interior. The amount of time spent by  $(B_s)$  in  $C$  during the interval  $[0, t]$  is  $\tau_C(t) = \int_0^t \mathbf{1}_C(B_s) ds$ .

**Proposition 3.5.** *We have a.s.  $\limsup_{t \rightarrow \infty} \frac{\tau_C(t)}{t} = 1$  and  $\liminf_{t \rightarrow \infty} \frac{\tau_C(t)}{t} = 0$ .*

*Proof.* Since the variable  $\limsup_{t \rightarrow \infty} \frac{\tau_C(t)}{t}$  is asymptotic, it is a.s. equal to a constant value  $\ell \in [0, 1]$ . Because of the scaling property of the Brownian motion and because  $C$  is a cone, we have

$$\begin{aligned} \mathbb{P}\left(\frac{\tau_C(t)}{t} \in I\right) &= \mathbb{P}\left(\left[\int_0^t \mathbf{1}_C(B_s) \frac{ds}{t}\right] \in I\right) = \mathbb{P}\left(\left[\int_0^1 \mathbf{1}_C(B_{ts}) ds\right] \in I\right) \\ &= \mathbb{P}\left(\left[\int_0^1 \mathbf{1}_C(\sqrt{t}B_s) ds\right] \in I\right) = \mathbb{P}\left(\left[\int_0^1 \mathbf{1}_C(B_s) ds\right] \in I\right). \end{aligned}$$

Take  $\alpha \in (0, 1)$ . As the cone  $C$  has a non empty interior, we have  $\mathbb{P}(B_\alpha \in \overset{\circ}{C}) > 0$  and, knowing that  $B_\alpha$  is in  $\overset{\circ}{C}$ , we also have  $\mathbb{P}(B_s \in \overset{\circ}{C}, \forall s \in (\alpha, 1)) > 0$ . The obvious

inequality

$$\mathbb{P}\left(\frac{\tau_C(t)}{t} > 1 - \alpha\right) \geq \mathbb{P}(B_s \in \overset{\circ}{C}, \forall s \in (\alpha, 1))$$

then implies

$$(15) \quad \mathbb{P}\left(\frac{\tau_C(t)}{t} > 1 - \alpha\right) > 0, \forall \alpha > 0.$$

We have  $\limsup_t \mathbb{P}\left(\frac{\tau_C(t)}{t} > \ell + \varepsilon\right) \leq \mathbb{P}(\limsup(\frac{\tau_C(t)}{t} > \ell + \varepsilon)) = 0, \forall \varepsilon > 0$ .

Now the distribution of  $\frac{\tau_C(t)}{t}$  does not depend on  $t$ , so that we have  $\mathbb{P}(\frac{\tau_C(t)}{t} > \ell + \varepsilon) = 0$ . In view of (15), this implies that  $\ell + \varepsilon > 1 - \alpha$ . But  $\alpha$  and  $\varepsilon$  being arbitrary small, one gets  $\ell \geq 1$ , that is  $\ell = 1$ . By considering the complement, we obtain the result for  $\liminf$ .  $\square$

This suggests that, if we can approximate our process  $(W_n^\Phi)$  by a Brownian motion, then the property claimed in Theorem 3.4 holds. This is the case, for example if we can assert that for every  $\gamma > 1/4$ , there exists  $C > 0$ , so that, for all  $t \in [0, 1]$ , one has a.s.

$$\|B(nt) - W_n^\Phi(t)\| \leq Cn^\gamma.$$

Such a property is sometimes called an almost sure invariance principle. It has been established for some hyperbolic or quasi-hyperbolic systems (see Gouëzel [Go10]). To deduce the desired property for  $W_n$  from the one satisfied by the Brownian motion, we need to control the amount of time spent by the Brownian motion not too far from the origin and to enlarge or shrink the cone we are interested in to get convenient estimates. We will not do these computations here because they are very similar to what is done below.

Indeed, we will show that the "plain" Donsker's invariance principle suffices. From the preceding proof for the Brownian motion, we just keep in mind that (15) in Proposition 3.5 is true.

We need to know that, most of the time,  $W_n$  is far from the origin:

**Lemma 3.6.** *If  $\Phi$  is not a coboundary, for every  $M > 0$ , the asymptotic frequency of visits of the process  $(W_n)_{n \geq 1}$  to the ball  $B(0, M)$  with center at the origin and radius  $M > 0$  in  $\mathbb{R}^d$  is almost surely zero:*

$$(16) \quad \lim_n \int_0^1 \mathbf{1}_{B(0, M)}(W_n(x, s)) ds = 0, \text{ for a.e. } x.$$

*Proof.* For  $K > 0$ , the ergodic theorem applied to  $(X \times \mathbb{R}^d, T_\Phi, \lambda = \mu \times dy)$  and  $\mathbf{1}_{B(0, K)}$  ensures the existence for a.e.  $(x, y) \in X \times \mathbb{R}^d$  of the limit

$$u_K(x, y) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{B(0, K)}(\Phi_k(x) + y).$$

The function  $u_K$  is integrable on  $X \times \mathbb{R}^d$ , nonnegative and  $T_\Phi$ -invariant. Suppose that  $u_K \neq 0$  on a set of positive measure. Then  $u_K \lambda$  is a finite  $T_\Phi$ -invariant measure on  $X \times \mathbb{R}^d$ , absolutely continuous with respect to  $\lambda$ . Since  $(X, T, \mu)$  is ergodic,

this implies that  $\Phi$  is a coboundary [Co79], contrary to the assumption. Therefore  $u_K = 0$  a.e. for the measure  $\lambda$ .

Taking  $K = M + 1$ , since  $\mathbf{1}_{B(0,M+1)}(\Phi_k(x) + y) \geq \mathbf{1}_{B(0,M)}(\Phi_k(x))$ , for  $\|y\| \leq 1$ , this implies:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{B(0,M)}(\Phi_k(x)) = 0, \text{ for a.e. } x.$$

Now we compare the discrete sum with the integral:

$$\int_0^1 \mathbf{1}_{B(0,M)}(W_n(x, s)) ds = \frac{1}{n} \sum_{k=0}^{n-1} \int_k^{k+1} \mathbf{1}_{B(0,M)}(\Phi_k(x) + (t-k)\Phi(T^k x)) dt.$$

Let  $\varepsilon > 0$  be arbitrary and  $K$  be such that  $\mu(|\Phi| > K) \leq \varepsilon$ . We have for  $t \in [k, k+1]$ :

$$\mathbf{1}_{B(0,M)}(\Phi_k(x) + (t-k)\Phi(T^k x)) \leq \mathbf{1}_{|\Phi(T^k x)| > K} + \mathbf{1}_{B(0,M+K)}(\Phi_k(x)),$$

so that for a.e.  $x$ .

$$\begin{aligned} & \limsup_n \int_0^1 \mathbf{1}_{B(0,M)}(W_n(x, s)) ds \\ & \leq \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{B(0,M+K)}(\Phi_k(x)) + \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{(|\Phi| > K)}(T^k x) \leq 0 + \varepsilon. \end{aligned}$$

It implies (16).  $\square$

**Notation 3.7.** Let us take  $a > 0$  and  $\underline{u}$  in the unit sphere in  $\mathbb{R}^d$ . For  $a$  and  $\underline{u}$  fixed, for every  $t > 0$  we denote by  $C_{a,t}$  or simply  $C_t$  the cone  $\{\underline{v} \in \mathbb{R}^d : \|\frac{\underline{v}}{\|\underline{v}\|} - \underline{u}\| < at\}$ .

**Lemma 3.8.** *Suppose  $\Phi$  is not a coboundary. The set of discontinuity points of the increasing function  $t \mapsto \limsup_{n \rightarrow \infty} \tau_{n,C_t}(x)$  is a.e. constant (with respect to  $x$ ). If  $t$  does not belong to this set of discontinuity points, then  $\limsup_{n \rightarrow \infty} \tau_{n,C_t}(x)$  is almost surely constant in  $x$ .*

*Proof.* Let us compare  $\tau_{n,C_t}(x)$  and  $\tau_{n,C_t}(Tx)$ . Take  $\varepsilon > 0$ . For every  $k \geq 1$ , we have  $\Phi_k(Tx) = \Phi_k(x) - \Phi(x) + \Phi(T^k x)$  and  $\|\Phi_k(Tx) - \Phi_k(x)\| \leq 2\|\Phi\|_\infty$ . There is  $M$  such that, if  $\Phi_k(x) > M$  and  $\Phi_k(x) \in C_t$ , then  $\Phi_k(Tx) \in C_{t+\varepsilon}$ .

Therefore, for  $s \in [0, 1]$ , we have  $W_n(Tx, s) \in C_{t+\varepsilon}$ , when  $W_n(x, s) \in C_t$  and  $W_n(Tx, s) \geq M$ . This implies:

$$\begin{aligned} \int_0^1 \mathbf{1}_{C_t}(W_n(x, s)) ds &= \int_0^1 \mathbf{1}_{C_t \cap B(0,M)}(W_n(x, s)) ds + \int_0^1 \mathbf{1}_{C_t \cap B(0,M)^c}(W_n(x, s)) ds \\ &\leq \int_0^1 \mathbf{1}_{B(0,M)}(W_n(x, s)) ds + \int_0^1 \mathbf{1}_{C_{t+\varepsilon}}(W_n(Tx, s)) ds. \end{aligned}$$

When  $n$  tends to infinity, the first integral tends to 0 almost surely by (16) (Lemma 3.6) if  $\Phi$  is not a coboundary. It follows:

$$\limsup_n \tau_{n,C_t}(x) \leq \limsup_n \tau_{n,C_{t+\varepsilon}}(Tx).$$

In the same way, we have  $\limsup_n \tau_{n,C_t}(Tx) \leq \limsup_n \tau_{n,C_{t+\varepsilon}}(x)$ . It follows, for every positive real numbers  $s < t < u < v$ :

$$\limsup_{n \rightarrow \infty} \tau_{n,C_s}(x) \leq \limsup_{n \rightarrow \infty} \tau_{n,C_t}(Tx) \leq \limsup_{n \rightarrow \infty} \tau_{n,C_u}(x) \leq \limsup_{n \rightarrow \infty} \tau_{n,C_v}(Tx).$$

This implies  $\lim_{s \rightarrow t, s > t} \limsup_{n \rightarrow \infty} \tau_{n,C_s}(x) = \lim_{s \rightarrow t, s > t} \limsup_{n \rightarrow \infty} \tau_{n,C_s}(Tx)$ , for every  $t$ .

Thus the common limit defines a map from  $X$  into the set of increasing functions on  $[0, 1]$ . This map is  $T$ -invariant, hence almost surely constant by ergodicity of  $T$ . In particular the finite or countable set of discontinuity points of  $t \mapsto \limsup_{n \rightarrow \infty} \tau_{n,C_t}(x)$  is independent of  $x$ . Outside this at most countable set of values of  $t$ ,  $\limsup_{n \rightarrow \infty} \tau_{n,C_t}(x)$  does not depend on  $x$ .  $\square$

Let  $A_0(C) := \{\chi \in \mathcal{C}_d([0, 1]) : \int_0^1 \mathbf{1}_{\partial C}(\chi(s)) ds > 0\}$  be the set of functions taking their values in the boundary of  $C$  for a set of positive measure of the variable  $s$ .

**Lemma 3.9.** *If the Lebesgue measure of the boundary of  $C$  is zero, then the Wiener measure of the set  $A_0(C)$  is 0.*

*Proof.* The Wiener measure of  $A_0(C)$  is

$$W(A_0(C)) = \mathbb{P}((B_\cdot) \in A_0(C)) = \mathbb{P}(\{\omega : \int_0^1 \mathbf{1}_{\partial C}(B_s(\omega)) ds > 0\}).$$

From the assumption on  $C$ , we have  $\mathbb{P}(B_s \in \partial C) = 0$  for every  $s$  and therefore  $\mathbb{E}\left(\int_0^1 \mathbf{1}_{\partial C}(B_s) ds\right) = \int_0^1 \mathbb{P}(B_s \in \partial C) ds = 0$ .  $\square$

**Remark 1.** It is clear that the set  $\Delta$  of atoms of the distribution of  $\int_0^1 \mathbf{1}_C(B_s) ds$  (the image probability on  $[0, 1]$  of the Wiener measure on  $\mathcal{C}_d([0, 1])$ ) is at most countable. If  $c \notin \Delta$ , then the set of functions  $\chi$  in  $\mathcal{C}_d([0, 1])$  such that  $\int_0^1 \mathbf{1}_C(\chi(s)) ds = c$  has zero measure for the Wiener measure.

For  $\eta > 0$ , denote by  $\partial C(\eta)$  the set of points in  $\mathbb{R}^d$  at a distance  $\leq \eta$  from the boundary of  $C$ .

**Proof of Theorem 3.4:** Since the interior of  $C$  is non empty, there is a family of cones  $C_t = C_{a,t}$  contained in  $C$ , constructed like the cones introduced in the preceding lemmas. We take a real  $t > 0$  such that  $\limsup_{n \rightarrow \infty} \tau_{n,C_t}(x)$  is almost surely constant in  $x$ . If we show that  $\limsup_{n \rightarrow \infty} \tau_{n,C_t} = 1$ , then also  $\limsup_{n \rightarrow \infty} \tau_{n,C} = 1$ .

From now, on we replace  $C$  by  $C_t$  still denoted  $C$ . In particular, the boundary of  $C$  now has Lebesgue measure 0. The invariance principle " $W_n(\cdot)/\sqrt{n} \rightarrow B$ ." means that, for every continuous functional  $F$  on  $\mathcal{C}_d([0, 1])$ , we have

$$(17) \quad \mathbb{E}\left(F\left(\frac{W_n(*, \cdot)}{\sqrt{n}}\right)\right) \rightarrow \mathbb{E}(F(B)).$$

Suppose that a sequence of probability measures  $(\mathbb{P}_n)$  defined on a space  $X$  converges weakly to  $\mathbb{P}$ . By Theorem 2.7 in Billingsley's book [Bi], if a measurable function  $\Psi$

from  $X$  to a metric space  $Y$  has a set of discontinuity points of measure zero for  $\mathbb{P}$ , then the sequence of pushforward measures  $(\mathbb{P}_{n,\Psi})$  converges to the pushforward measure  $\mathbb{P}_\psi$ .

For  $\Psi$  we take here the function  $F_c$ ,  $c > 0$ , defined on the metric space of continuous functions from  $[0, 1]$  to  $\mathbb{R}^d$  with the uniform norm by

$$(18) \quad F_c(\chi) = \mathbf{1}_{[c, +\infty[} \left( \int_0^1 \mathbf{1}_C(\chi(s)) ds \right).$$

In order to apply to  $F_c$  the quoted theorem and the convergence (17), we have to show that the set of discontinuity points of  $F_c$  has measure zero for the Wiener measure.

Assume that  $c \notin \Delta$  (i.e.,  $c$  is not an atom of the distribution of  $\int_0^1 \mathbf{1}_C(B_s) ds$ ). Let us consider the set  $\mathcal{G}_c$  of functions  $\chi_0$  such that  $\chi_0 \notin A_0(C)$  (i.e.  $\{t : \chi_0(t) \in \partial C\}$  has Lebesgue measure 0) and  $\int_0^1 \mathbf{1}_C(\chi_0(s)) ds \neq c$ . It has full Wiener measure by Lemma 3.9 and Remark 1.

Let us show that  $F_c$  is continuous on the set  $\mathcal{G}_c$ .

Let  $\varepsilon$  be such that  $0 < \varepsilon < |\int_0^1 \mathbf{1}_C(\chi_0(s)) ds - c|$ . The measure of the set of times  $s$  for which  $\chi_0(s)$  is at a distance less than  $\eta$  from the boundary of  $C$  tends to 0 when  $\eta$  tends to 0. We can take  $\eta > 0$  such that this measure is less than  $\varepsilon$ .

Let  $\chi$  be another function at a uniform distance less than  $\eta$  from  $\chi_0$ . If  $\chi_0(s)$  is not in  $\partial C(\eta)$ , then  $\chi_0(s)$  and  $\chi(s)$  are either both in  $C^c$  or both in  $C$ . Thus, we have

$$\begin{aligned} & \left| \int_0^1 \mathbf{1}_C(\chi_0(s)) ds - \int_0^1 \mathbf{1}_C(\chi(s)) ds \right| \\ & \leq \left[ \int_0^1 \mathbf{1}_{\partial C(\eta)}(\chi_0(s) + \mathbf{1}_{(\partial C(\eta))^c}(\chi_0(s))) |\mathbf{1}_C(\chi_0(s)) - \mathbf{1}_C(\chi(s))| ds \right. \\ & \left. \leq \int_0^1 \mathbf{1}_{\partial C(\eta)}(\chi_0(s)) ds + \int_0^1 \mathbf{1}_{(\partial C(\eta))^c}(\chi_0(s)) |\mathbf{1}_C(\chi_0(s)) - \mathbf{1}_C(\chi(s))| ds \leq \varepsilon + 0. \right. \end{aligned}$$

Therefore:  $\mathbf{1}_{[c, +\infty[} \left( \int_0^1 \mathbf{1}_C(\chi(s)) ds \right) = \mathbf{1}_{[c, +\infty[} \left( \int_0^1 \mathbf{1}_C(\chi_0(s)) ds \right)$  and we have proved that the functional  $F_c$  is continuous at  $\chi_0$ .

Finally we have shown, for every  $c \notin \Delta$ , the continuity of  $F_c$  on the set  $\mathcal{G}_c$  which has full Wiener measure.

For  $c$  outside  $\Delta$  (which is at most countable), it follows from the theorem mentioned above:

$$\mathbb{E} \left( \frac{F_c(W_n(*, \cdot))}{\sqrt{n}} \right) \rightarrow \mathbb{E}(F_c(B)),$$

that is:

$$(19) \quad \lim_n \mathbb{P} \left( \left[ \int_0^1 \mathbf{1}_C \left( \frac{W_n(x, s)}{\sqrt{n}} \right) ds \right] \geq c \right) = \mathbb{P} \left( \left[ \int_0^1 \mathbf{1}_C(B_s) ds \right] \geq c \right).$$

As  $C$  is a cone, we have:

$$\int_0^1 \mathbf{1}_C\left(\frac{W_n(x, s)}{\sqrt{n}}\right) ds = \tau_{n,C}(x).$$

Because of (15), (19) implies that, for every  $c < 1$  with  $c \notin \Delta$ ,  $\mathbb{P}(\tau_{n,C} \geq c) > 0$  for  $n$  large enough. As a consequence, we have

$$\mathbb{P}(\limsup_n \tau_{n,C} \geq c) \geq \limsup_n \mathbb{P}(\tau_{n,C} \geq c) > 0.$$

Hence,  $\limsup_n \tau_{n,C}$  being constant, it follows  $\limsup_n \tau_{n,C} \geq 1$ . As  $\tau_{n,C} \in [0, 1]$ , this proves  $\limsup_n \tau_{n,C} = 1$ .  $\square$

**Remark 2.** We can also consider the piecewise constant function:  $V_n(x, s) := \Phi_k(x)$  for  $s \in [k/n, (k+1)/n[$ . If  $\Phi$  is bounded, then

$$\|W_n(x, \cdot) - V_n(x, \cdot)\|_\infty \leq \|\Phi\|_\infty.$$

On the other hand, we have

$$\int_0^1 \mathbf{1}_C(V_n(x, s)) ds = \frac{1}{n} \int_0^n \mathbf{1}_C(V_n(x, \frac{t}{n})) dt = \frac{1}{n} \text{Card}\{k \leq n : \Phi_k(x) \in C\}.$$

Reasoning as before we can show that if a cone  $C$  contains a cone of the form  $C_t$ , like in Lemma 3.8, then

$$\limsup_n \int_0^1 \mathbf{1}_C(V_n(x, s)) ds \geq \limsup_n \int_0^1 \mathbf{1}_{C_t}(W_n(x, s)) ds = 1.$$

This means that, if  $\Phi$  is a bounded function satisfying Donsker's invariance principle, we also have the following discrete version of the property claimed in the theorem:

$$\limsup_n \frac{1}{n} \text{Card}\{k \leq n : \Phi_k(x) \in C\} = 1.$$

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